PENANG FREE SCHOOL (2005)
MATHEMATICS T (PAPER 1)

1. Express $\dfrac{3+2x}{(2-x)(3+x^2)}$ in partial fractions. [6]

2. Find the value of m and n if $x^4 + 12x^3 + + 46x^2 + mx + n$ is the square of a quadratic expression. [6]

3. Integrate (i) $\displaystyle\int \frac{2}{\sin 2\theta}\, d\theta$ (ii) $\displaystyle\int (\ln x)^2\, dx$ [6]

4. Given that $y = \ln(\sin^3 2x)$, prove that $3\dfrac{d^2y}{dx^2} + \left(\dfrac{dy}{dx}\right)^2 + 36 = 0$. [6]

5. (i) If $e^x - e^{-x} = 4$, evaluate $e^x + e^{-x}$.

   (ii) Solve $(\ln x)^{\ln x} = x^2$. [7]

6. Using the matrix method, solve the following equations :

   $x - 3y - a = 0$, $2x + z - b = 0$, $4x + y + 3z - c = 0$,

   expressing x, y and z in terms of a, b and c. [7]

7. Determine the turning points on the curve $y = e^x \cos x$.
   Distinguish between maxima and minima. [7]

8. Find the equation of the tangents to the circle $x^2 + y^2 - 12x - 14y + 75 = 0$ which are parallel to the line $3y - x = 0$. [7]

9. (i) Determine the shortest distance from the line $4y - 3x = 5$ to the parabola $y^2 = 4x - 4$.

   (ii) Any chord of a parabola passing through the focus is called a focal chord. Find the locus of the mid-point of focal chords of the parabola $y^2 = 4ax$. [10]

10. (i) If z is a complex number, prove that $zz^* = |z|^2$.

    (ii) Find the values of c and d such that $(c + di)^2 = i$.
    Hence, solve the equation $z^2 + 2z + 1 - i = 0$.

    (iii) If α and β are roots of the equation $(1 + i)z^2 - 2iz + (3 + i) = 0$, find the quadratic equation whose roots are $\alpha + 3\beta$ and $3\alpha + \beta$. [12]

11. (i) Evaluate $(10.02)^{10}$ to the nearest thousand, using the binomial theorem.

    (ii) Expand $(1 - x)^{-2}$ in ascending powers of x.

    Deduce that $\displaystyle\sum_{n=0}^{\infty} (a + bn)x^n = \frac{a + (b-a)x}{(1-x)^2}$, $|x| < 1$.

    Hence, evaluate $\displaystyle\sum_{n=0}^{\infty} \frac{8n+1}{3^{2n}}$ [12]

12. (i) Find the coordinates of the points of intersection of the curve $y = x^3 - 4x$ with the line $y = 3x + 6$. Hence, solve the inequality $x^3 - 4x > 3x + 6$.

    (ii) Find the equation of the normal to the curve above at the point (2, 0).

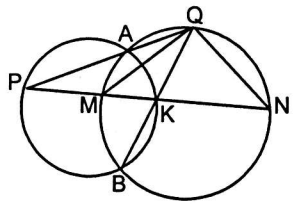
    (iii) Show graphically, that the equation $x^3 - 4x - 5 = 0$ has only one real root.

    Taking $x = 2.5$ as a first approximation to this root, use the Newton-Raphson method to find a second approximation, giving your answer correct to 2 decimal places. [14]

MATHEMATICS T (PAPER 2)

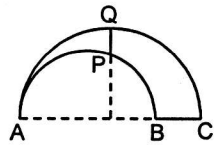
1. Express $y = \sin\theta - 2\cos\theta$ in the form of $R\sin(\theta - \alpha)$.
   Hence find the maximum and minimum values of y. [4]

2. Solve the equation $\sin(x + 30^\circ) + \sin 2x + \sin(x - 30^\circ)$, given that $0^\circ < x < 360^\circ$.
   Give your answers correct to one decimal places. [4]

3. 

   Given that PAQ, QKB and PMKN are straight lines. Prove that QM = QN. [4]

LEE HOCK LEONG
B. Sc (Ed.) - 1st. Class Hons.
M. Sc (Pure Maths.) - CGPA4

4. 

   Given that the circles touch at A, QP produced cuts AB at right angle at the centre of the larger circle. If PQ = 3 cm and BC = 5 cm, find the radius of each circle. [4]

5. (a) Verify that $y = x - \frac{1}{x}$ is a solution to the differential equation $x\frac{dy}{dx} + y = 2x$. [4]

(b) Given that the differential equation $\frac{dP}{dt} = 0.08P(1 - \frac{P}{1000})$ and $P(0) = 100$.

Show that $P(t) = \frac{1000}{1+9e^{-0.08t}}$. Hence, find the value of t for which $P(t) = 100$. [8]

6. The points $A(2, 0)$, $B(0, 3)$ and the variable point $C(h,1)$ moves in the x–y plane parallel to the y–axis. By considering the dot product of the vectors $\overrightarrow{AC}$ and $\overrightarrow{BC}$, find the possible values of h when the angle of ACB is 90°. [6]

7. At time $t = 0$, the point P moves from the origin with speed $\sqrt{80}$ ms^{-1} along the line whose equation is $y = \frac{x}{2}$. Find the velocity of P in the form of the vector.

If the point Q starts at the same time from the point (8, 0) and moves with speed 5 ms^{-1}, what are the equations of the lines that the point Q must move in order to meet P? Find the time taken in each case. [6]

8 (a) The numbers below are the times I took to travel to school on ten consecutive days, given to the nearest minutes :

36, 34, 31, 29, 40, 37, 35, 34, 39,35

Calculate the mean and standard deviation of these times.

If the time taken for my journey is normally distributed with the mean and variance of the above data, what is the probability that my journey takes longer than 37 minutes?

How much time should I allow myself for travelling to school if I will accept being late one day in ten?

(b) In an agriculture experiment the gain in mass, in kilograms, of 100 pigs during a certain period were recorded as follow :

Gain in mass (kg)	5 – 9	10 – 14	15 – 19	20 – 24	25 – 29	30 – 34
Frequency	2	29	37	16	14	2

Calculate

(i) the mode

(ii) the semi-interquartile range

(iii) the mean and standard deviation. [6]

9. (a) In a public speaking contest there are three finalists and three judges. Assuming that the judges choose independently and that each is equally likely to select any of the three finalists as the winner, find the probability that

(i) all will choose the same winner.

(ii) all will choose different winner.

(iii) two will be alike and the third different. [4]

(b) A bag contains two balls. It is equally likely that the bag contains 0, 1 or 2 black balls. If a ball is drawn at random and found to be black, what is the probability that the other is also black? [4]

(c) If p, q, r and s are positive integers such that $8 < p + q + r + s < 11$, how many different solutions are possible? [4]

10. (a) A small firm owns five cars. On average the weekday (Monday – Friday) demand is two cars per day, and the demand is three cars per day during the weekend (Saturday and Sunday).
Find the probability of having to refuse a customer on

(i) a weekday. (ii) both days during a weekend. [6]

(b) In a large town, one person in 80, on the average has blood type X. If 200 person are taken at random and Y is the number of person having blood type X, state the distribution of Y.

Using (i) the binomial distribution (ii) Poisson approximation, find how many donors must be taken at random in order that the probability of including at least one donor of blood type X shall be 0.9 or more? [6]

11. (i) Suppose X is uniformly distributed over the interval $-1 \le x \le 1$, find

(a) $P(X^2 \le \frac{1}{4})$ (ii) $P(X > \frac{3}{4} \mid |X| > \frac{1}{2})$ [4]

(ii) The probability density function is given by $f(x) = \begin{cases} k(ax - x^2) & ,0 \le x \le 2 \\ 0 & ,\text{otherwise} \end{cases}$,

where k and a are positive constants. Show that $a \ge 2$ and $k = \frac{3}{6a-8}$.

Given that the mean value of x is 1, calculate the values of a and k.

For these values of a and k, sketch the graph of the probability density function, and find the variance. [8]

12. (i) My postman delivers mail on Monday at time T which is normally distributed with mean 8.50 am and with variance 225 minutes^2. Find the probability that

(a) the postman arrives after I have left for work at 9 am. [3]

(b) the postman arrives between 7.40 am and 8.55 am. [3]

(ii) A fair die is tossed 120 times. If the random variable X denotes the number of times the face 6 turns up,

(a) state the distribution of X, giving the parameters [1]

(b) write the formula for $P(X = r)$, where $r = 0, 1, 2, 3, \ldots, 120$. [1]

(c) estimate $P(X \le 14)$. [4]

LEE HOCK LEONG
B. Sc (Ed.) - 1st. Class Hons.
M. Sc (Pure Maths.) - CGPA4

Penang Free School (2005)

No:..... MT1 Date:.....

1. Let $f(x) = \dfrac{3+2x}{(2-x)(3+x^2)}$

$\equiv \dfrac{A}{2-x} + \dfrac{Bx+C}{3+x^2} = \dfrac{A(3+x^2)+(Bx+C)(2-x)}{(2-x)(3+x^2)}$

$\therefore\ 3+2x \equiv A(3+x^2)+(Bx+C)(2-x)$

$x=2$: $3+4 = A(7) \Rightarrow A=1$

$[x^2]$: $0 = A-B \Rightarrow B=1$

$[x^0]$: $3 = 3A+2C \Rightarrow C=0$

$\therefore\ f(x) = \dfrac{1}{2-x} + \dfrac{x}{3+x^2}$ #

2. Let $x^4+12x^3+46x^2+mx+n$

$\equiv (x^2+kx+p)^2 = x^4+2kx^3+(2p+k^2)x^2+2pkx+p^2$

$[x^3]$: $12 = 2k \Rightarrow k=6$

$[x^2]$: $46 = 2p+k^2 \Rightarrow p=5 \ \therefore n=25$

$[x]$: $m = 2pk = 2(5)(6) = 60$ #

3. (i) $\displaystyle\int \frac{2}{\sin 2\theta}\,d\theta = \int \frac{2(s^2+c^2)}{2sc}\,d\theta$

$= \displaystyle\int \left[\frac{\sin\theta}{\cos\theta} + \frac{\cos\theta}{\sin\theta}\right]d\theta$

$= -\ln|\cos\theta| + \ln|\sin\theta| + c$

$= \ln|\tan\theta| + c$ #

(ii) $\displaystyle\int (\ln x)^2\,dx = x(\ln x)^2 - \int x(2)\frac{\ln x}{x}\,dx$

$= x(\ln x)^2 - \left[2x\ln x - \int 2x\left(\frac{1}{x}\right)dx\right]$

$= x(\ln x)^2 - 2x\ln x + 2x + c$ #

4. $y = \ln(\sin^3 2x) = 3\ln\sin 2x$

$e^{\frac{y}{3}} = \sin 2x \Rightarrow \frac{1}{3}e^{\frac{y}{3}}\frac{dy}{dx} = 2\cos 2x$

$\therefore\ \frac{1}{9}e^{\frac{y}{3}}\left(\frac{dy}{dx}\right)^2 + \frac{1}{3}e^{\frac{y}{3}}\frac{d^2y}{dx^2} = -4\sin 2x$

$\therefore\ 3\dfrac{d^2y}{dx^2} + \left(\dfrac{dy}{dx}\right)^2 + 36 = 0$ #$\ [\because e^{\frac{y}{3}} = \sin 2x]$

5 (i) Let $e^x + e^{-x} = k$: $e^x - e^{-x} = 4$

$\therefore\ 2e^x = k+4,\ 2e^{-x} = k-4$

$(k-4)(k+4) = 4 \Rightarrow k^2 = 20$

$\therefore\ k = 2\sqrt{5}$ #$\ [\because e^x > 0 \Rightarrow k > 0]$

(ii) $(\ln x)^{\ln x} = x^2 \Rightarrow \ln x[\ln(\ln x)] = 2\ln x$

$\therefore\ \ln x[\ln(\ln x) - 2] = 0$

$\therefore\ \ln x = 0$ or $\ln(\ln x) = 2$

$\therefore\ x = e^{e^2}$ #$\ [\because \ln x \neq 0]$

6. $\left.\begin{array}{l} x-3y-a=0 \\ 2x+z-b=0 \\ 4x+y+3z-c=0 \end{array}\right\}$ $\begin{pmatrix} 1 & -3 & 0 \\ 2 & 0 & 1 \\ 4 & 1 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$

Let $A = \begin{pmatrix} 1 & -3 & 0 \\ 2 & 0 & 1 \\ 4 & 1 & 3 \end{pmatrix}$: $(C_{ij}) = \begin{pmatrix} -1 & -2 & 2 \\ 9 & 3 & -13 \\ -3 & -1 & 6 \end{pmatrix}$

$\therefore\ |A| = 1(-1)+(-3)(-2)+0 = 5$

$\therefore\ \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \frac{1}{5}\begin{pmatrix} -1 & 9 & -3 \\ -2 & 3 & -1 \\ 2 & -13 & 6 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix}$

$\therefore\ x = \frac{1}{5}(-a+9b-3c)$, $y = \frac{1}{5}(-2a+3b-c)$

$z = \frac{1}{5}(2a-13b+6c)$ #

7. $y = e^x\cos x \Rightarrow \dfrac{dy}{dx} = e^x(-\sin x) + e^x\cos x$

$\dfrac{d^2y}{dx^2} = \dfrac{dy}{dx} + e^x[-\cos x - \sin x]$

For t.points: $\dfrac{dy}{dx} = 0 \Rightarrow -\sin x + \cos x = 0$

$\therefore\ \tan x = 1 \Rightarrow x = n\pi + \frac{\pi}{4}\ (n \in \mathbb{Z})$

$\therefore$ Turning points: $(n\pi + \frac{\pi}{4}, e^{n\pi+\frac{\pi}{4}})$ #

$n = 2k\ (k \in \mathbb{Z})$: $\dfrac{d^2y}{dx^2} < 0$

$n = 2k+1\ (k \in \mathbb{Z})$: $\dfrac{d^2y}{dx^2} > 0$ ↗ min.

$\therefore\ (2k\pi + \frac{\pi}{4}, e^{2k\pi+\frac{\pi}{4}})_{max}$, $((2k+1)\pi + \frac{\pi}{4}, e^{(2k+1)\pi+\frac{\pi}{4}})$ #

8. $x^2+y^2-12x-14y+75 = 0$

$\therefore\ (x-6)^2+(y-7)^2 = 10 \Rightarrow C(6,7), r=\sqrt{10}$

Let tangent: $3y-x+k=0$

$\therefore\ d_c = \sqrt{10}$: $\dfrac{|3(7)-(6)+k|}{\sqrt{9+1}} = \sqrt{10}$

$\therefore\ |k+15| = 10 \Rightarrow k+15 = \pm 10$

$\therefore\ k = -5$ or -25

$\therefore$ tangents: $3y-x-5=0$, $3y-x-25=0$ #

9. (i) Let $P(p^2+1, 2p)$ on $y^2 = 4(x-1)$

$d_P = \dfrac{|4(2p)-3(p^2+1)-3|}{(4^2+3^2)^{1/2}} = \frac{2}{5}\left|p^2 - \frac{8}{3}p + \frac{8}{3}\right|$

$= \frac{2}{5}\left|(p-\frac{4}{3})^2 + \frac{8}{9}\right|$

$\therefore\ (d_P)_{min} = \frac{2}{5}\left|\frac{8}{9}\right| = \frac{8}{15}$ #

(ii) focus: $S(a,0)$, Let $P(ap^2, 2ap)$

$Q(aq^2, 2aq)$ on parabola and focal chord.

$\therefore\ m_{PS} = m_{PQ}$: $\dfrac{2ap-0}{ap^2-a} = \dfrac{2ap-2aq}{ap^2-aq^2}$

$\therefore\ \dfrac{p}{p^2-1} = \dfrac{1}{p+q} \Rightarrow pq = -1$

Let $M(x,y) = \left(\dfrac{ap^2+aq^2}{2}, ap+aq\right)$

$\therefore\ 2x = a(p^2+q^2)$ and $y = a(p+q)$

$\therefore\ 2x = a[(p+q)^2 - 2pq]$

$\therefore\ 2x = a[(\frac{y}{a})^2 - 2(-1)]$

$\therefore\ y^2 = 2a(x-1)$ #$\ [$locus of M$]$

LEE HOCK LEONG
B. Sc (Ed.) - 1st. Class Hons.
M. Sc (Pure Maths.) - CGPA4

10. (i) let $z = a+bi$: $(z^* = a-bi)$

$\therefore zz^* = a^2 - b^2(-1) = a^2+b^2 = |z|^2$ #

(ii) $(c+di)^2 = i \Rightarrow c^2-d^2+2cdi = i$

$\therefore c^2-d^2=0 \Rightarrow c=\pm d$, and $2cd = 1$

$c=d$: $2d^2=1 \Rightarrow d = \pm\frac{1}{\sqrt2} = c$

$c=-d$: $-2d^2=1 \Rightarrow d \notin \mathbb{R}$.

$\therefore z^2+2z+1-i=0 \Rightarrow (z+1)^2 = i$

$\therefore z+1 = \pm\frac{1}{\sqrt2}(1+i)$

$\therefore z = \frac{1}{\sqrt2}-1+\frac{1}{\sqrt2}i$ or $-\frac{1}{\sqrt2}-1-\frac{1}{\sqrt2}i$ #

(iii) $(1+i)z^2-2iz+(3+i)=0$ (roots $=\alpha,\beta$)

$\therefore \alpha+\beta = \frac{2i}{1+i} = 1+i$, $\alpha\beta = \frac{3+i}{1+i} = 2-i$

Roots $= \alpha+3\beta,\ 3\alpha+\beta$:

Sum $= 4(\alpha+\beta) = 4(1+i)$

Product $= 3(\alpha^2+\beta^2)+10\alpha\beta$

$= 3(\alpha+\beta)^2+4\alpha\beta = 3(1+i)^2+4(2-i)$

$= 8+2i$

$\therefore$ QE: $x^2-4(1+i)x+(8+2i)=0$ #

11. (i) $(10.02)^{10} = (10+0.02)^{10}$

$= 10^{10}+10(10)^9(0.02)+45(10)^8(0.02)^2$

$+120(10)^7(0.02)^3+210(10)^6(0.02)^4+\dots$

$= 10201810000$ # (nearest thousand)

(ii) $(1-x)^{-2} = 1+(-2)(-x)+\frac{(-2)(-3)}{2!}(-x)^2+\dots$

$= 1+2x+3x^2+\dots+(n+1)x^n+\dots$

$= \sum_{n=0}^{\infty}(n+1)x^n$ #

$\sum_{n=0}^{\infty}(a+bn)x^n = \sum_{n=0}^{\infty}[a(1+n)x^n+(bn-an)x^n]$

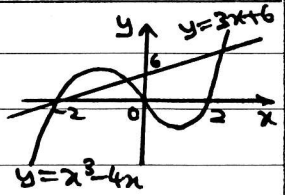
$= \frac{a}{(1-x)^2}+(b-a)x[1+2x+3x^2+\dots]$

$= \frac{a+(b-a)x}{(1-x)^2}$ #, $|x|<1$ (VE: $|x|<1$)

$a=1, b=8, x=\frac{1}{9}$: $\sum_{n=0}^{\infty}(1+8n)(\frac{1}{9})^n$

$= \sum_{n=0}^{\infty}\frac{8n+1}{3^{2n}} = \frac{1+7(1/3)}{(2/3)^2} = \frac{15}{2}$ #

12. (i) $x^3-4x = 3x+6$



$\therefore x(x-2)(x+2) = 3(x+2)$

$\therefore (x+2)[x^2-2x-3]=0$

$\therefore (x+2)(x+1)(x-3)=0$

$\therefore x=-2,-1$ or $3 \Rightarrow$ intersection points $(-2,0), (-1,3), (3,15)$ #

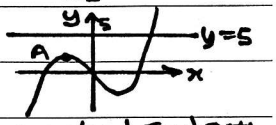
For $x^3-4x>3x+6$: $\{x: -2<x<-1$ or $3<x\}$ #

(ii) $y = x^3-4x \Rightarrow \frac{dy}{dx} = 3x^2-4$

Normal at $(2,0)$: $\frac{y-0}{x-2} = \frac{-1}{12-4} \Rightarrow 8y+x=2$ #

(iii) For $\frac{dy}{dx}=0$: $x^2=\frac{4}{3} \Rightarrow x = \pm\frac{2}{\sqrt3}$

$\therefore A(-\frac{2}{\sqrt3}, \frac{16}{\sqrt3})$ max.



$y=x^3-4x$ and $y=5$ intersects once only $\Rightarrow x^3-4x-5=0$ has only 1 real root #

Let $f(x) = x^3-4x-5$; $f'(x) = 3x^2-4$

$x=2.5$: $x_1 = 2.5 - \frac{f(2.5)}{f'(2.5)}$

$\therefore x_1 = 2.5 - \frac{0.625}{14.75} = 2.46$ (2 dp) #

1. $y = \sin\theta - 2\cos\theta = R\sin(\theta-\alpha)$

$= R\sin\theta\cos\alpha - R\cos\theta\sin\alpha$

$\therefore R\cos\alpha = 1$ and $\therefore \tan\alpha = 2$

$R\sin\alpha = 2$ $\therefore \alpha = \tan^{-1}(2)$

$\therefore R(\frac{2}{\sqrt5}) = 2 \Rightarrow R = \sqrt5$

$\therefore y = \sqrt5\sin[\theta-\tan^{-1}(2)]$

$\therefore y_{max} = \sqrt5$ # (since $-1\le\sin A\le1$)

$y_{min} = -\sqrt5$ #

2. $\sin(x+30°)+\sin 2x+\sin(x-30°)=0$

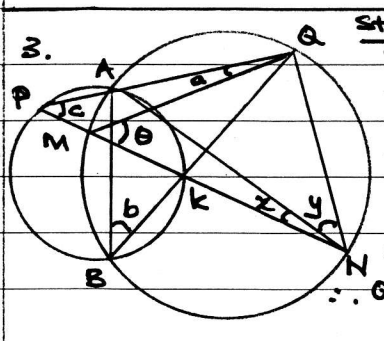
$\therefore 2\sin x\cos x + 2\sin x\cos 30° = 0$

$\therefore \sin x(2\cos x+\sqrt3)=0$

$\therefore \sin x = 0$ or $\cos x = -\frac{\sqrt3}{2}$

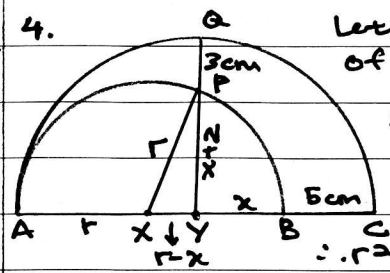
$\therefore x = 150°, 180°, 210°$ #

3.



Statements	Reasons
$x=a$	Angles in the same segment
$y=b$	"
$b=c$	"
$\theta=a+c$	Exterior angle of ΔPMQ
$\therefore \theta = x+y$	
$\therefore QM=QN$ #	ΔQMN is isosceles

4.



Let X, Y are centres of the circles.

$AY = YC$

$2r-x = x+5$

$\therefore 2r = 2x+5$

ΔXYP: $r^2 = (x+2)^2+(r-x)^2$

$\therefore r^2 = x^2+4x+4+r^2+x^2-2rx$

$\therefore 2x^2+4x+4-x(2x+5)=0 \Rightarrow x=4$

$\therefore r_{big} = 4+5 = 9$cm #, $r_{small} = r = 6.5$cm #

5.(a) $y = x - \frac{1}{x} \Rightarrow xy = x^2 - 1$ —$(\frac{d}{dx})$

$\therefore x\frac{dy}{dx} + y = 2x$ —— ①

$\therefore y = x - \frac{1}{x}$ is a solution to ① ✱

(FMath method): $\frac{dy}{dx} + \frac{y}{x} = 2$

Integral Factor $= e^{\int \frac{1}{x}dx} = e^{\ln x} = x$

$\therefore x\frac{dy}{dx} + y = 2x \Rightarrow \frac{d}{dx}(xy) = 2x$

$\therefore xy = x^2 + c \Rightarrow y = x + \frac{c}{x}$

Let $c = -1$: $y = x - \frac{1}{x}$ is a solution to ①

(b) $\frac{dP}{dt} = 0.08P(1 - \frac{P}{1000})$

$\therefore \int_{100}^{P} \frac{1000}{P(1000-P)}dP = \int_0^t 0.08\,dt$

$\therefore \int_{100}^{P} [\frac{1}{1000-P} + \frac{1}{P}] = 0.08[t]_0^t$

$\therefore [\ln(\frac{P}{1000-P})]_{100}^{P} = 0.08t$

$\therefore \ln[\frac{9P}{1000-P}] = 0.08t \Rightarrow \frac{1000}{9P} - \frac{1}{9} = e^{-0.08t}$

$\therefore P = \frac{1000}{1+9e^{-0.08t}}$ ✱

when $P = 900$: $0.08t = \ln 81 \Rightarrow t = 50\ln 3$ ✱

6. $\angle ACB = 90°$: $\overrightarrow{AC}\cdot\overrightarrow{BC} = 0 \Rightarrow \begin{pmatrix} h-2 \\ 1 \end{pmatrix}\cdot\begin{pmatrix} h \\ -2 \end{pmatrix} = 0$

$\therefore h^2 - 2h - 2 = 0 \Rightarrow h = \frac{2 \pm \sqrt{4+8}}{2}$

$\therefore h = 1 \pm \sqrt{3}$ ✱

7. $v_P = \sqrt{80}[\frac{1}{\sqrt{5}}\begin{pmatrix} 2 \\ 1 \end{pmatrix}] = \begin{pmatrix} 8 \\ 4 \end{pmatrix}$ ✱

To meet: $S_P = S_Q$: $t\begin{pmatrix} 8 \\ 4 \end{pmatrix} = \begin{pmatrix} 8 \\ 0 \end{pmatrix} + t\begin{pmatrix} p \\ q \end{pmatrix}$

$\therefore \left.\begin{matrix} 8t = 8 + tp \\ 4t = tq \end{matrix}\right\} \therefore q = 4$

$|\begin{pmatrix} p \\ q \end{pmatrix}| = 5$: $p^2 + 4^2 = 5^2 \Rightarrow p = \pm 3$

$p = 3$: Eq: $y = \frac{4}{3}(x-8)$, $t = \frac{8}{5}$s ✱

$p = -3$: Eq: $y = -\frac{4}{3}(x-8)$, $t = \frac{8}{11}$s ✱

8.(a) $n = 10$, $\Sigma x = 350$, $\Sigma x^2 = 12350$

$\therefore \bar{x} = \frac{350}{10} = 35$ ✱, $s = \sqrt{\frac{12350}{10} - 35^2} = \sqrt{10}$ ✱

$P(X > 37) = P(Z > \frac{37-35}{\sqrt{10}}) = 0.2635$ ✱

$P(X > k) = \frac{1}{10}$: $P(Z > \frac{k-35}{\sqrt{10}}) = 0.1$

$\therefore k - 35 = \sqrt{10}(1.282) \Rightarrow k = 39.05$

Time allow $\approx$ 39 minutes. ✱

(b)(i) mode $= 14.5 + (\frac{8}{8+21})(5) = 15.88$ ✱

(ii) $SIQR = ([19.5 + \frac{7}{16}(5)] - [9.5 + \frac{23}{34}(5)]) \times \frac{1}{2} = 4.11$ ✱

(iii) $n = 100$, $\Sigma fx = 1785$, $\Sigma fx^2 = 34965$

$\therefore \bar{x} = \frac{1785}{100} = 17.85$ ✱

$\therefore s = \sqrt{\frac{34965}{100} - 17.85^2} = 5.57$ ✱

9.(a)(i) Prob. $= \frac{1}{3}(\frac{1}{3})(\frac{1}{3})(3) = \frac{1}{9}$ ✱

(ii) Prob $= \frac{1}{3}(\frac{2}{3})(\frac{1}{3})(3) = \frac{2}{9}$ ✱

(iii) Prob $= 1 - (\frac{1}{9} + \frac{2}{9}) = \frac{1}{3}$ ✱

(b) Let X = no. of black ball, $P_0 = P_1 = P_2 = \frac{1}{3}$

$P(X=2 | X \geq 1) = \frac{P(X=2)}{P(X \geq 1)} = \frac{1}{3}/2(\frac{1}{3}) = \frac{1}{2}$ ✱

(c) Let $A = p+q+r+s$:

$A = 9$: (1116) (1125) (1134) (1224) (1233) (2223)

$A = 10$: (1117) (1126) (1135) (1144) (1225) (1234) (2224) (2233)

$\therefore$ No. of solutions $= 4! + \frac{4!}{2!}(7) + \frac{4!}{3!}(4) + \frac{4!}{2!2!}(2) = 136$ ✱

10 (a) Let X = no. of demands (weekday, $X \sim P_0(2)$)

Y = no. of demands (weekend), $Y \sim P_0(3)$

(i) $P(X > 5) = 1 - e^{-2}(1 + 2 + \frac{2^2}{2!} + \frac{2^3}{3!} + \frac{2^4}{4!} + \frac{2^5}{5!})$

$= 0.01656$ ✱

(ii) $P(Y > 5)^2 = [1 - e^{-3}(1 + 3 + \frac{3^2}{2!} + \frac{3^3}{3!} + \frac{3^4}{4!} + \frac{3^5}{5!})]^2$

$= 0.00704$ ✱

(b) $Y \sim B(200, \frac{1}{80})$ ✱ (Let W = no. of X)

(i) $P(W \geq 1) \geq 0.9$: $1 - P_0 \geq 0.9$ ($W \sim B(n, \frac{1}{80})$)

$\therefore (\frac{79}{80})^n \leq 0.1 \Rightarrow n\lg(\frac{79}{80}) \leq -1$

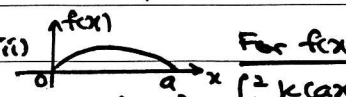
$\therefore n \geq 183.05 \Rightarrow n_{min} = 184$ ✱

(ii) $P(W \geq 1) \geq 0.9$: $1 - e^{-\frac{n}{80}} \geq 0.9$ ($W \sim P_0(\frac{n}{80})$)

$\therefore -\frac{n}{80} \leq \ln 0.1 \Rightarrow n \geq 184.2$ $\therefore n_{min} = 185$ ✱

11. (i) (a) $P(X^2 \leq \frac{1}{4}) = P(-\frac{1}{2} \leq X \leq \frac{1}{2}) = 1(\frac{1}{2}) = \frac{1}{2}$ ✱

(b) $P(X > \frac{3}{4} | |X| > \frac{1}{2}) = \frac{P(X > \frac{3}{4})}{2P(X > \frac{1}{2})} = \frac{\frac{1}{4}(\frac{1}{4})}{2(\frac{1}{2})(\frac{1}{4})} = \frac{1}{4}$ ✱

(ii) 

For $f(x) \geq 0$ in $0 \leq x \leq 2$: $a \geq 2$ ✱

$\int_0^2 k(ax - x^2)dx = 1 = k(\frac{ax^2}{2} - \frac{x^3}{3})$

$\therefore 1 = k[2a - \frac{8}{3}] \Rightarrow k = \frac{3}{6a-8}$. ✱

$E(X) = 1$: $\int_0^2 k[ax^2 - x^3]dx = 1 = k[\frac{ax^3}{3} - \frac{x^4}{4}]_0^2$

$\therefore \frac{3}{6a-8}[\frac{8a}{3} - 4] = 1 \Rightarrow a = 2$ $\therefore k = \frac{3}{4}$ ✱

$E(X^2) = \int_0^2 \frac{3}{4}[2x^3 - x^4]dx = \frac{3}{4}[\frac{x^4}{2} - \frac{x^5}{5}]_0^2$

$E(X^2) = \frac{6}{5}$: $\therefore Var(X) = \frac{6}{5} - 1^2 = \frac{1}{5}$ ✱

12.(i) (a) $P(T > 9.00) = P(Z > \frac{10}{15}) = 0.2525$ ✱

(b) $P(7.40 < T < 8.55) = P(-\frac{10}{15} < Z < \frac{5}{10})$

$= P(Z > -\frac{2}{3}) - P(Z > 0.5) = 0.7475 - 0.3085 = 0.439$ ✱

12.(ii) (a) $X \sim B(120, \frac{1}{6})$ ✱

(b) $P(X = r) = {}^{120}C_r(\frac{1}{6})^r(\frac{5}{6})^{120-r}$ ✱

(c) $X \sim N(20, \frac{50}{3})$

$P(X \leq 14) \rightarrow P(X < 14.5)$

$= P(Z < \frac{-5.5}{\sqrt{50/3}}) = 0.0890$ ✱

Uni